Conformal Quantum Mechanics and Sine-Square Deformation†

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In Refs. 1,2), it was shown that sine-square deformation (SSD)[3)] for two-dimensional (2D) conformal field theory (CFT)[4)] can be understood by introducing a new quantization scheme called "dipolar quantization." The basic idea was generalized in Ref. 5) to incorporate the entanglement Hamiltonian and other interesting deformations of 1D CFT. Here, we examine whether the idea of dipolar quantization is applicable to the one-dimensional (1D) case, which is called conformal quantum mechanics (CQM) and was first studied in the seminal paper by de Alfaro, Fubini, and Furlan.[6)]

CQM can be realized by the following Lagrangian:

$$L = \frac{1}{2}(\dot{q}(t))^2 - \frac{g}{2}\frac{1}{q(t)^2}, \tag{1}$$

where t is the 1D "spacetime" coordinate. The Lagrangian Eq. (1) is invariant under the following transformations:

$$t \to t' = \frac{at+b}{ct+d}, \quad ad - bc = 1, \tag{2}$$

$$q(t) \to q'(t') = \frac{1}{ct+d}q(t), \tag{3}$$

which constitute 1D conformal transformations.

The transformation Eq. (2) for t can be conveniently decomposed into the following three components:
Translation, with $a = d = 1$ and $c = 0$, giving

$$t \to t + b. \tag{4}$$

Dilatation, with $a = 1/d$ and $b = c = 0$, giving

$$t \to a^2 t. \tag{5}$$

Special conformal transformation (SCT), with $a = d = 1$ and $b = 0$, giving

$$t \to \frac{t}{ct+1}. \tag{6}$$

The above transformations are generated by the following operators, respectively:

$$H = P_0 = \frac{1}{2}p(t)^2 + \frac{g}{2}\frac{1}{q(t)^2}, \tag{7}$$

$$D = -\frac{1}{4}\left(p(t)q(t) + q(t)p(t)\right), \tag{8}$$

$$K_0 = \frac{1}{2}q(t)^2. \tag{9}$$

In Ref. 6), de Alfaro, Fubini, and Furlan introduced the new operator

$$R \equiv \frac{1}{2}\left(aP_0 + \frac{1}{a}K_0\right), \tag{10}$$

and proposed to regard it, instead of H, as the time-translation operator, *i.e.*, the Hamiltonian. Here, we introduce another operator

$$\bar{R} \equiv H - K = \frac{1}{2}p(t)^2 + \frac{1}{2}\frac{g}{q(t)^2} - \frac{1}{2}q(t)^2, \tag{11}$$

and showed that it corresponds to the thermal density operator. Therefore, $\bar{R}$ is reminiscent of the entanglement Hamiltonian in 2D CFT case.

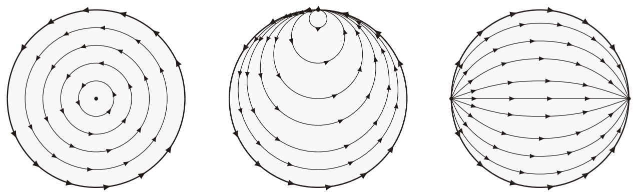

Fig. 1. Time translation on the Poincaré disk. On the boundary of the disk (thick line), "time flow" is uniform without a fixed point for R, while it is limited to the finite region bounded by the two fixed points for $\bar{R}$. H exhibits marginal behavior, and it has one fixed point at infinity. The connection to dipolar quantization is apparent in this depiction.

As for SSD, from the symmetry consideration, we identify the original Hamiltonian $H = P_0$ as the SSD Hamiltonian. Figure 1 depicts the time flow generated by these three operators on the Poincaré disk.

References
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